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Tactical Memoranda I-III.

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INTRODUCTION.

Cayley* has divided Algebra into Tactic and Logistic. A tactical entity of a finite number (m) of letters $a_1 \ldots . a_m$ I call an *arrangement of degree* m. Every arrangement of degree m determines a substitution-group on the m letters G^m under which it is invariant. Two arrangements of the same degree which differ only by the notation of their letters belong to the same *class* of arrangements.

Under the general heading *Tactical Memoranda* I shall publish a series of papers on certain more or less closely connected topics of Tactic.

I.

THE GENERAL TACTICAL CONFIGURATION: DEFINITION AND NOTATION.

The technical term *configuration* was introduced—into projective geometry of the plane and of space—by Mr. Reye in his "Geometrie der Lage," second edition, vol. 1, p. 4, 1876, as stated in his memoir, "Das Problem der Configurationen" (Acta Mathematica, vol. 1, pp. 92–96, 1882). Particular geometric configurations, occurring in the plane, in space, or, in general in flat space of n dimensions, have been investigated by (among others) Messrs. De Vries, Jung, S. Kantor, Klein, Martinetti, Reye, Schoenflies, Segre, and Veronese. I have however come across no explicit formulation of the definition of the general geometric configuration occurring in flat space of n dimensions. As soon as one undertakes to construct such a formulation, he comes inevitably to the definition and matrix-notation to be at once given. *This matrix-notation should*

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*Cayley, "On the Notion and Boundaries of Algebra" (Quarterly Journal of Mathematics, vol. 6, 1864; "Collected Mathematical Papers," vol. 5 [347]).

then be accepted as the definitive general notation for geometric configurations, while, of course, as heretofore, for particular investigations, other particular notations may be found satisfactory.

Definition of and notation for the general tactical configuration of rank n.

We have at disposal n sets of objects (letters), a_1 in the first set, a_2 in the second set, and, in general, a_i in the i^{th} set; we denote these objects by

$$\lambda_{ij^{(i)}} \qquad \begin{pmatrix} i = 1, 2, \dots n \\ j^{(i)} = 1, 2, \dots a_i \end{pmatrix}$$

where the first suffix picks out the set to which the object belongs, and the second picks out the object in that set.

We have at disposal further a certain relation—call it *incidence*[*]—between certain objects of *different* sets, so that we can say that $\lambda_{i_1j_1}$ is or is not incident with $\lambda_{i_2j_2}$ $(i_1 \neq i_2)$. We tabulate these incidences occurring in the system of objects in a *table of incidences* of which one line is

$$\lambda_{i_1j_1}\,[\lambda_{i'_1j'_1}, \lambda_{i'_2j'_2}, \dots \lambda_{i'_kj'_k} \dots],$$

which is read, $\lambda_{i_1j_1}$ is incident with $\lambda_{i'_kj'_k}$ $(k = 1, 2, \dots)$.

This table of incidences, when arranged so as to show also the distribution of the objects into the n sets, furnishes the complete tactical definition of our system of n sets of objects subject to the otherwise undefined incidence-relation.

This system is called a *tactical configuration of rank n*, if for every g, h $(g \neq h)$ every object of the set g is incident with the same number, say a_{gh}, of objects of the set h. I inscribe the fundamental numbers a_i, a_{gh} of a configuration of rank n in the square *matrix-symbol*

$$(a_{gh}) \qquad \begin{matrix} (g, h = 1, 2 \dots n) \\ (a_{gg} = a_g) \end{matrix}$$

which symbol serves likewise as *the notation* for the configuration. The g^{th} element of the principal diagonal of the matrix $a_{gg} = a_g$ is the number of objects of the g^{th} set. For an element a_{gh} $(g \neq h)$ not on the principal diagonal, the matrix is read *in by the line* (g) *and out by the column* (h), thus—Every object of the g^{th} set is incident with a_{gh} objects of the h^{th} set.

The *group of the configuration* is the totality of substitutions leaving the configuration invariant, that is, permuting amongst themselves the (objects) letters of every same set and leaving invariant the table of incidences of the configuration. This group has the degree $a_1 + a_2 + \ldots . + a_n$, say a, and will be denoted by G^{a}_{Cf}. The group G^{a}_{Cf} is intransitive, and is isomorphic with the n groups $G^{a_i}_{Cf}$ ($i = 1, 2, \ldots . n$) on the letters of the several sets; the substitutions of G^{a}_{Cf} quâ substitutions on the a_i letters $\lambda_{ij^{(i)}}$ of set i are the substitutions (perhaps with repetitions) of the $G^{a_i}_{Cf}$. Those configurations will be of most interest whose groups are of most interest. It is convenient* to transfer many terms from the abstract and the substitution group theories to the configuration theory; e. g. the Cf is l-ply transitive on the letters of set i if the $G^{a_i}_{Cf}$ is l-ply transitive.

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[*] The term *incidence* is technical, having in general tactical theory no closer definition. The sequel will make this point of view clearer.

Particular types of configurations are reached by *particularizing the incidence-relation.* We may for instance postulate that the incidence-relation shall be (a) *reciprocal* and (b) *partly transitive,* that is, (a) whenever $\lambda_{i_1j_1}$ is incident with $\lambda_{i_2j_2}$, then $\lambda_{i_2j_2}$ is incident with $\lambda_{i_1j_1}$; (b) there is a certain (open) succession of the n sets which, by a proper choice of the notation, we take to be the succession $i = 1, 2, \ldots . n$, such that whenever $\lambda_{i_1j_1}$ is incident with $\lambda_{i_2j_2}$ and $\lambda_{i_2j_2}$ in its turn is incident with $\lambda_{i_3j_3}$ where either $i_1 > i_2 > i_3$ or $i_1 < i_2 < i_3$, then $\lambda_{i_1j_1}$ is incident with $\lambda_{i_3j_3}$. We then reach a general type of configurations which includes all geometrical configurations (at least as ordinarily treated) and which we therefore call the *geometrico-tactical* configurations. Of course we reach the geometric configurations proper—of rank n—by identifying the objects of the set i with geometric flat spaces of $i - 1$ dimensions R_{i-1} lying in flat space of n dimensions R_n, and identifying the incidence-relation with the geometric incidence (the relation of *containing or being contained in*), which fulfils the postulates (a), (b). Obviously by postulating the properties (a) and (b) separately, and the latter indeed in either or both of its two forms, we reach more general types of configurations than the geometrico-tactical type.

I have elsewhere† indicated certain so-called linear configurations of rank n (any integer) which will serve as interesting examples of geometrico-tactical configurations.

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*Mr. Schoenflies, in his memoir "Ueber die regelmässigen Configurationen n_3" (Mathematische Annalen, vol. 31, pp. 43-69, 1888), calls (loc. cit., p. 44) a geometric configuration *regular* whose group is transitive on the *Cf-points.* It is, however, always desirable not to introduce new terms if without confusion old terms can be employed in new senses suggested naturally by the old senses.

† Moore, "Concerning Jordan's Linear Groups" (Bulletin of the American Mathematical Society, November, 1895).

II.

TACTICAL SYSTEMS

$$S[k,\ l,\ m],\ S\{k,\ l,\ m\},\ SGS[k,\ l,\ m],\ SGS\{k,\ l,\ m\},$$
$$S[\underset{\cdot}{k},\ l,\ m],\ S\{\underset{\cdot}{k},\ l,\ m\},\quad \textit{and}\quad SGS\{\underset{\cdot}{k},\ l,\ m\}.$$

1. *Introductory definitions.* We deal with m distinct letters $a_1 \ldots\ldots a_m$. A k-ad is a combination of k of the m letters $[a_{i_1} \ldots\ldots a_{i_k}]$, the order being immaterial. A $\underset{\cdot}{k}$-id is an arrangement of k of the m letters in a definite order $\{a_{i_1} \ldots\ldots a_{i_k}\}$. In a $\underset{\cdot}{k}$-ad (or $\underset{\cdot}{k}$-id) the k letters may or may not be distinct. A k-ad (or k-id) is a $\underset{\cdot}{k}$-ad (or $\underset{\cdot}{k}$-id) with k distinct letters. In m letters there are altogether m_k[*] k-ids, m_k $\underset{\cdot}{k}$-ids, m_k/k_k k-ads, and $m_k^{\dagger}/k_k$[†] $\underset{\cdot}{k}$-ads.

If the m elements were (instead of letters) m distinct *positions*, similar definitions would hold for the *positional* k-ad, k-id. [We do not now need the positional $\underset{\cdot}{k}$-ad, $\underset{\cdot}{k}$-id.]

If from a $\underset{\cdot}{k}$-ad we select, without regard to order, certain l letters ($l \leqq k$), taking no letter oftener than it occurs in the $\underset{\cdot}{k}$-ad, we have an $\underset{\cdot}{l}$-ad. A $\underset{\cdot}{k}$-ad and an $\underset{\cdot}{l}$-ad so related are called *incident* each to the other. An $\underset{\cdot}{l}$-ad incident to each of two $\underset{\cdot}{k}$-ads is *common* to the two $\underset{\cdot}{k}$-ads.

If from a $\underset{\cdot}{k}$-id we select in order certain l letters ($l \leqq k$) from distinct positions on the face of the $\underset{\cdot}{k}$-id, we have an $\underset{\cdot}{l}$-id, and say that the $\underset{\cdot}{l}$-id is *incident* with the $\underset{\cdot}{k}$-id and *occupies* a certain *l-idic position on the face of the $\underset{\cdot}{k}$-id.* A $\underset{\cdot}{k}$-id has k_l incident $\underset{\cdot}{l}$-ids; a k-id has k_l *distinct* incident l-ids. An $\underset{\cdot}{l}$-id incident to each of two $\underset{\cdot}{k}$-ids and occupying corresponding l-idic positions on their faces is *common* to the two $\underset{\cdot}{k}$-ids.

2. $S[k,\ l,\ m]$. *A k-adic system*[‡] *of index l in m letters* $S[k,\ l,\ m]$ $(m \geqq k \geqq l)$ is an arrangement of the m letters in k-ads in such a way that every l-ad of the m letters is incident with one and only one k-ad of the system $S[k,\ l,\ m]$. The *characters* k, l, m are the *class* k, the *index* l and the *degree* m of the $S[k,\ l,\ m]$.

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[*] r being a positive integer, the notations n_r, $n_r^{\dagger}$ are thus defined:

$$n_r = n(n-1)\ldots\ldots(n-r+1),$$
$$n_r^{\dagger} = n(n+1)\ldots\ldots(n+r-1).$$

[†] Smith's "Treatise on Algebra," 1st ed., p. 284.

[‡] The overburdened term *system* is retained because the *notion* $S[k,\ l,\ m]$ is a generalization of the well-known notion, *triple system* $\Delta_m \equiv S[3,\ 2,\ m]$.

A $S[m, l, m]$ is simply one m-ad. A $S[k, k, m]$ is simply the totality of k-ads of m letters. A $S[k, 1, m]$ exists for and only for $m = kv$. A $S[k, 1, kv]$ is an arrangement of the kv letters in v k-ads.

3. $S\{k, l, m\}$. *A k-idic system of index l in m letters* $S\{k, l, m\}$ $(m \geqq k \geqq l)$ is an arrangement of the m letters in k-ids in such a way that every l-id of the m letters appears on the faces of the k-ids of the system $S\{k, l, m\}$ k_l times, occupying every possible position once and only once. A $S\{k, k, m\}$ is simply the totality of k-ids of m letters. There is but one $S\{m, m-1, m\}$, viz., the $S\{m, m, m\}$.

4. *Theorems concerning a* $S[k, l, m]$.

(a). The number of k-ads in a $S[k, l, m]$ is m_l / k_l.

(b). If in a $S[k, l, m]$ we suppress all k-ads not containing a certain g-ad $(g < l)$ and then everywhere in the remaining k-ads suppress that g-ad, we have a $S[k-g, l-g, m-g]$ containing (a) $(m-g)_{l-g} / (k-g)_{l-g}$ $(k-g)$-ads.

(c). If the positive integers k, l, m $(m > k > l)$ are characters of a $S[k, l, m]$, then the l expressions

$$\frac{(m-g)_{l-g}}{(k-g)_{l-g}} = \frac{(m-l+1)_h^+}{(k-l+1)_h^+} \qquad (g+h=l) \qquad \begin{matrix}(g = 0, 1, \ldots . l-1)\\(h = 1, 2, \ldots . l)\end{matrix}$$

are positive integers (a, b). (These conditions are satisfied at once if $m = k$ or if $k = l$.)

(d). Setting $m - l = \mu$ and $k - l = d$, the form of $m = \mu + l$ with respect to $k = l + d$ and l, where the k, l, m satisfy the conditions (c), viz. that $\mu > d > 0$ and that the l expressions

$$(\mu + 1)_h^+ / (d+1)_h^+ \qquad\qquad (h = 1, 2, \ldots . l)$$

are positive integers, depends upon the form of μ with respect to the primes entering the l integers $d+1$, $d+2$, ... $d+l = k$.

(e). A general (but not exhaustive)* solution of the conditions (c) or (d) is the following, depending on the theory of the distribution of multiples of a prime in a succession of consecutive positive integers:

Let p_i $(i = 1, 2, \ldots . r)$ be the primes $\leqq d$ and q_j $(j = 1, 2, \ldots . s)$ be the primes $\geqq d+1, \leqq d+l = k$. Determine the exponents e_i $(i = 1, 2, \ldots . r)$ by the inequalities $p_i^{e_i-1} < d+1 \leqq p_i^{e_i}$ (whence $p_i^{e_i} \leqq d^2$). Determine for every prime p_i a set of numbers π_{it} $(t = 1, 2, \ldots . e_i)$ thus: π_{it} is the smallest integer $\geqq d+1$ which is divisible by p_i^t (so that $\pi_{it} \leqq \pi_{ie_i} = p_i^{e_i} \leqq d^2$).

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* Compare in the sequel, §8 c.

Then $m = \mu + l$ satisfies for given l and $k = l + d$ $(d \geqq 0)$ the conditions (c) or (d) if μ $(\mu > d)$ is modulo M congruent to no one of the integers $0, 1, 2, \ldots . d-1$, where M is in turn every one of the integers $\pi_{it} \begin{pmatrix} t = 1, 2, \ldots . e_i \\ i = 1, 2, \ldots . r \end{pmatrix}$, q_j $(j = 1, 2, \ldots . s)$, except however those π_{it} which exceed $k = l + d$. (There are no such excepted π_{it} if $k = l + d \geqq d^2$.)

(e′). These conditions (e) as regards the primes q are *necessary*.

(f). The solution (e) is for $d = 1$ and for $d = 2$ exhaustive.

($\mathrm{f}_{d=1}$). The conditions (c) or (d) on $m = \mu + l$, l and $k = l + 1$ are satisfied if and only if μ is relatively prime to $Q = \prod_{j=1}^{s} q_j$, or, what is the same thing, to $k!$. There are thus $\phi(Q)$ forms[*] of $m = \mu + l$ (modulo Q).

($\mathrm{f}_{d=2}$). The conditions (c) or (d) on $m = \mu + l$, l and $k = l + 2$ are satisfied if and only if $\mu \not\equiv 0, 1$ (mod. M) where M is in turn 4 and every prime q of the series $3, 4, \ldots . k$.

(g). Every $S[k, l, m]$ contains (many) $S[k-g, l-g, m-g]$ $(g = 1, 2, \ldots . l-1)$ (b). A $S[k, l, m]$ not lying thus in any including $S[k+1, l+1, m+1]$—and hence not thus in any including $S[k+f, l+f, m+f]$ $(f = 1, 2, \ldots .)$—is called a *maximum* $S[k, l, m]$. Every $S[k+f, l+f, m+f]$ $(f = 0, 1, \ldots .)$ containing a given $S[k, l, m]$, for which latter system the product $\mu_d = \mu(\mu-1) \ldots . (\mu - d + 1)$ $(\mu = m - l,\ d = k - l)$, contains a prime $q' \geqq d+1$, and hence (e′) $q' > k$ has its $f < k - q'$ (e′). Hence follows for every such existing $S[k, l, m]$ the *existence* of at least one containing *maximum* $S[k+f, l+f, m+f]$ $(0 \leqq f < k - q')$.

(h). *Every* $S[k, l, m]$ *may be looked at as a configuration*[†] *of rank* k. The objects of the i^{th} *Cf*-set $(i = 1, 2, \ldots . k)$ are the i-ads entering the k-ads of the $S[k, l, m]$.

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[*] To specialize for triple systems $\Delta_m \equiv S[3, 2, m] \equiv S[k, l, m]$; $m = \mu + 2$, $\mu \equiv 1, 5$ (mod. 6), $\therefore$ $m = 6\mu' + 1$ or $6\mu' + 3$, which is as it should be.

[†] That every triple system $\Delta_m \equiv S[3, 2, m]$ may be looked at as a configuration $Cf\begin{pmatrix} m & \frac{1}{2}(m-1) \\ 3 & \frac{1}{6}m(m-1) \end{pmatrix}$ was noted by Mr. De Vries, "Zur Theorie der Tripelsysteme" (Rendiconti del Circolo Matematico di Palermo, vol. 8, pp. 222–226, 1894).

(i). $(m-l)/(k-l)$ being an integer, $(m-l)/(k-l)$ $S[k, l, m]$, no two of whose k-ads have an $(l+1)$-ad in common, taken together constitute a $S[k, l+1, m]$.

5. *Theorems concerning a* $S\{k, l, m\}$.

(a). In m letters there are in all m_l l-ids. There are in all k_l l-idic positions on the face of a k-id. Every k-id has thus k_l distinct incident l-ids, and every l-id is incident with k_l distinct k-ids. Thus the $S\{k, l, m\}$ contains certain m_l k-ids connected with the (totality of) m_l l-ids by a configuration

$$\begin{pmatrix} m_l & k_l \\ k_l & m_l \end{pmatrix}.$$

(b). If in a $S\{k, l, m\}$ we suppress all k-ids not containing a certain g-id $(g < l)$ in a certain position and then everywhere in the remaining k-ids suppress that g-id, we have a $S\{k-g, l-g, m-g\}$. Hence arises (as in §4 g) the notion: *maximum* $S\{k, l, m\}$.

(c). If in a $S\{k, l, m\}$ we suppress in every k-id the g-id $(g \leqq k-l)$ occupying a certain g-idic position on the face of the k-id, we have left a $S\{k-g, l, m\}$.

(d). The process (c) gives us, if $k \geqq 2l$, $g \geqq l$, $\therefore$ $k-g \geqq l$, two systems $S\{k-g, l, m\}$, $S\{g, l, m\}$ with an interesting $1-1$ correspondence between the $(k-g)$-ids of one and the g-ids of the other.

(e). The process (c) gives us, if $g < l$ and $g \leqq k-l$, a $S\{k-g, l, m\}$ containing m^g $S\{k-g, l-g, m-g\}$.

(f). $m-l$ $S\{k, l, m\}$, no two of whose k-ids have an $(l+1)$-id in common, taken together constitute a $S\{k, l+1, m\}$.

6. $SGS[k, l, m]$. *A k-adic school-girl-system** *of index l in m letters* $SGS[k, l, m]$ $(m \geqq k \geqq l)$ is a k-adic system of index l in m letters $S[k, l, m]$ whose m_l/k_l k-ads are collected into sets $((s_1))$, every k-ad entering one and only one set s_1, and every set s_1 containing in its k-ads every one of the m letters, i. e.

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* This arrangement is a generalization of the *fifteen-school-girls* arrangement $SGS[3, 2, 15]$. See for instance Cayley, "On a Tactical Theorem relating to the Triads of Fifteen Things" (Collected Mathematical Papers, vol. 5 [323]); Power, "On the Problem of the Fifteen School-girls" (Quarterly Journal of Mathematics, vol. 8, 1867).

every 1-ad of the m letters once and only once, and then these sets $((s_1))$ are similarly collected into sets $((s_2))$, every set s_1 entering one and only one set s_2, and every set s_2 containing in its k-ads every 2-ad of the m letters, and so on, ..., in general, the sets $((s_{g-1}))$ $(g = 1, 2, \ldots . l)$ are similarly collected into sets $((s_g))$, every set s_{g-1} entering one and only one set s_g, and every set s_g containing in its k-ads every g-ad of the m letters once and only once. Obviously the k-ads play the rôle of sets $((s_0))$ and the $S[k, l, m]$ that of *the single* set s_l. Every set s_g $(g = 1, 2, \ldots . l)$ is a $SGS[k, g, m]$. There is no distinction of order in the sets s_{g-1} lying in the same set s_g. A $S[m, l, m]$ is at once a $SGS[m, l, m]$. A $S[k, 1, m = kv]$ is at once a $SGS[k, 1, kv]$. The same $S[k, l, m]$ may underlie a number of different $SGS[k, l, m]$.

7. $SGS\{k, l, m\}$. *A k-idic school-girl-system of index l in m letters* $SGS\{k, l, m\}$ $(m \geqq k \geqq l)$ is a k-idic system of index l in m letters $S\{k, l, m\}$ whose m_l/k_l k-ids are collected into sets $((s_1))$, every k-id entering one and only one set s_1, and every set s_1 containing in its k-ids every one of the m letters, i. e. every 1-id of the m letters once and only once in every position, and then these sets $((s_1))$ are similarly collected into sets $((s_2))$, every set s_1 entering one and only one set s_2, and every set s_2 containing in its k-ids every 2-id of the m letters once and only once in every position, and so on, ..., and in general, the sets $((s_{g-1}))$ $(g = 1, 2, \ldots . l)$ are similarly collected into sets $((s_g))$, every set s_{g-1} entering one and only one set s_g, and every set s_g containing in its k-ids every g-id of the m letters once and only once in every position. Obviously the k-ids play the rôle of sets $((s_0))$ and the $SGS\{k, l, m\}$ that of *the single* set s_l. Every set s_g is a $SGS\{k, g, m\}$. The same $S\{k, l, m\}$ may underlie a number of different $SGS\{k, l, m\}$. Every $SGS\{m, m-1, m\}$ is also a $SGS\{m, m, m\}$.

8. *Theorems concerning a* $SGS[k, l, m]$.

(a). Every set s_g $(g = 0, 1, \ldots . l)$ of a $SGS[k, l, m]$ contains m_g/k_g k-ads. Every set s_h $(h > g)$ contains $(m_h/k_h) \div (m_h/k_h) = (m-g)_{h-g}/(k-g)_{h-g}$ sets s_g. The $SGS[k, l, m]$ contains $(m-g)_{l-g}/(k-g)_{l-g}$ sets s_g.

(b). Every set s_{g+1} contains $(m_{g+1}/k_{g+1}) \div (m_g/k_g) = (m-g)/(k-g)$ sets s_g $(g = 0, 1, \ldots . l-1)$.

(c). If the positive integers k, l, m $(m > k \geqq l)$ are the characters of a $SGS[k, l, m]$, then the l expressions

$$(m-g)/(k-g) \qquad (g = 0, 1, \ldots . l-1)$$

are positive integers greater than 1, that is, m has the form*

$$m = k(M\nu + 1),$$

where kM is the least common multiple of the l integers $k, k-1, \ldots. k-l+1$, and where ν is a positive integer. Of course we have here another† general (but not exhaustive) solution of the conditions of §4 c.

(d). If in a $SGS[k, l, m]$ we suppress all k-ads not containing a certain g-ad $(g < l)$ and then everywhere in the remaining k-ads suppress that g-ad, we have a $SGS[k-g, l-g, m-g]$, every set s_{f+g} $(f = 1, 2, \ldots. l-g)$ of the original system yielding a set s_f of the new system.

(e). Every $SGS[k, l, m]$ contains (many) $SGS[k-g, l-g, m-g]$ $(g = 1, 2, \ldots. l-1)$ (d). A $SGS[k, l, m]$ not lying thus in any including $SGS[k+1, l+1, m+1]$—and hence not thus in any including $SGS[k+f, l+f, m+f]$ $(f = 1, 2, \ldots.)$—is called a *maximum* $SGS[k, l, m]$. Every existent $SGS[k, l, m]$ lies in at least one including *maximum* $SGS[k+f, l+f, m+f]$, whose f satisfies the conditions $0 \leqq f \leqq f'$, where $k+1+f'$ $(f' \geqq 0)$ is the smallest integer greater than k which is not a divisor of $m-k$ (c).

(f). The $SGS[k, l, m]$ is a geometrico-tactical configuration of rank $l+1$. The $a_g = a_{gg}$ objects of set g $(g = 0, 1, \ldots. l)$ are the $a_g = a_{gg} = (m-g)_{l-g}/(k-g)_{l-g}$ sets s_g. The fundamental numbers

$$a_{gh}\,(g, h = 0, 1, \ldots. l;\ g \neq h)$$

are $$(g < h)\quad a_{gh} = 1;\quad (g > h)\quad a_{gh} = (m-h)_{g-h}/(k-h)_{g-h}.$$

(g). $(m-l)/(k-l)$ being an integer, $(m-l)/(k-l)$ $SGS[k, l, m]$, no two of whose k-ads have an $(l+1)$-ad in common, taken together constitute a $SGS[k, l+1, m]$.

9. *Theorems concerning a* $SGS\{k, l, m\}$.

(a). Every set s_g $(g = 0, 1, \ldots. l)$ of a $SGS\{k, l, m\}$ is a $S\{k, g, m\}$ and (§5 a) contains m_g k-ids. Every set s_h $(h > g)$ contains $m_h/m_g = (m-g)_{h-g}$ sets s_g. The $SGS\{k, l, m\}$ contains $(m-g)_{l-g}$ sets s_g.

(b). Every set s_{g+1} contains $m_{g+1}/m_g = m-g$ sets s_g $(g = 0, 1, \ldots. l)$.

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* These expressions $(m-g)/(k-g)$ have at once each the value 1 if $m = k$. The form $m = k(M\nu+1)$ holds then for all cases $m \geqq k \geqq l$ if we take $\nu \geqq 0$.

† Compare §4 e.

(c). The $SGS\{k, l, m\}$ is a geometrico-tactical configuration of rank $l+1$. The $a_g = a_{gg}$ objects of set g ($g = 0, 1, \ldots . l$) are the $a_g = a_{gg} = (m-g)_{l-g}$ sets s_g. The fundamental numbers a_{gh} ($g, h = 0, 1, \ldots . l$; $g \neq h$) are

$$(g < h) \quad a_{gh} = 1; \ (g > h) \quad a_{gh} = (m-h)_{g-h}.$$

(d). If in a $SGS\{k, l, m\}$ we suppress all k-ids not containing a certain g-id ($g < l$) in a certain position and then everywhere in the remaining k-ids suppress that g-id, we have a $SGS\{k-g, l-g, m-g\}$, every set s_{f+g} ($f = 1, 2, \ldots . l-g$) of the original system yielding a set s_f of the new system. Hence arises (as in §8 e) the notion: *maximum* $SGS\{k, l, m\}$.

(e, f, g, h). Theorems e, f, g, h for the $SGS\{k, l, m\}$ are analogues of theorems §5 c, d, e, f for the $S\{k, l, m\}$.

10. *Examples of* $SGS[k, l, m]$*—and so of* $S[k, l, m]$.

(a). $SGS[3, 2, 15]$. See the references above to papers on the original school-girl problem.

(b). $SGS[2, 2, 4]$: $[b_0 b_1][b_2 b_3]$, $[b_0 b_2][b_3 b_1]$, $[b_0 b_3][b_1 b_2]$. There is but one $SGS[2, 2, 4]$.

(c). $SGS[2, 2, 6]$: symbol $\infty.01234$ (which is to be thought of graphically as a regular pentagon 01234 and its center ∞). The six letters are $b_\infty b_0 b_1 b_2 b_3 b_4$, or writing merely the indices $\infty 01234$. One set s_1 of three duads of the $SGS[2, 2, 6]$ is $[\infty 0][14][23]$. From this, by the cyclic substitution on five indices (01234), we have in all five sets s_1. This $SGS[2, 2, 6]$ has a substitution-group G^6_{120} generated by the substitutions $(\infty)(01234)$, $(\infty)(0)(14)(23)$, $(\infty)(0)(1243)$, $(\infty 0)(14)(2)(3)$—(which may well be thought graphically on the symbol given for the $SGS[2, 2, 6]$)—which is the well-known G^6_{120} of degree 6 and of order 120, holoedrically isomorphic with the symmetric substitution-group on five letters $G^5_{5!}$. There are six such $SGS[2, 2, 6]$ conjugate under the symmetric $G^6_{6!}$. Every existent $SGS[2, 2, 6]$ belongs to this class of six.

(d). $SGS[2, 2, 8]$. Mr. Noether, in §1 of his memoir, "Ueber die Gleichungen achten Grades und ihr Auftreten in der Theorie der Curven vierter Ordnung" (Mathematische Annalen, vol. 15, pp. 89–110, 1879), sets out to make an exhaustive determination of all $SGS[2, 2, 8]$.

(e). *SGS* [3, 2, 9]. The nine letters b_{ij} are given by their indices ij $(i, j = 0, 1, 2)$. The symbol

$$\begin{array}{ccc} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{array}$$

implies that the *SGS* [3, 2, 9] consists of the following four sets s_1 of three triads each:

[00 01 02][10 11 12][20 21 22],
[00 10 20][01 11 21][02 12 22],
[00 11 22][01 12 20][02 10 21],
[00 12 21][01 10 22][02 10 21].

This *SGS* [3, 2, 9] defines the only existent class of *SGS* [3, 2, 9]. It occurs geometrically in the inflection-point configuration of a plane cubic curve. It occurs also in group-theory* as a particular case $(p = 3)$ of the more general system (f).

(f). *SGS* $[p, 2, p^n]$ (p a prime, n a positive integer), connected with the Abelian or commutative group G_{p^n} determined by n generators $A_1 \dots A_n$, each of period p. The p^n letters $b_{i_1 i_2 \dots i_n}$ of the *SGS* $[p, 2, p^n]$ denote the p^n elements $A_1^{i_1} A_2^{i_2} \dots A_n^{i_n}$ of the G_{p^n}, where the n indices or exponents $i_1 \dots i_n$ are integers taken modulo p. Every cyclic sub-group G_p determines one p-ad of the *SGS* $[p, 2, p^n]$, and indeed—when the G_{p^n} is written as a rectangular array with respect to this G_p as a sub-group—a set s_1 of p-ads of the *SGS* $[p, 2, p^n]$. (f) is a particular case $(n' = 1)$ of the more general system (g).

(g). *SGS* $[p^{n'}, 2, p^{nn'}]$ (p a prime, n, n' positive integers).

The *SGS* $[p, 2, p^n]$ (f) depends in the way explained upon the *abstract* Abelian group G_{p^n}. When, however, it is studied from the standpoint of a certain *concrete* Abelian G_{p^n}, viz. the Galois-field† of order p^n *GF* $[p^n]$ quâ additive-group, it is seen to depend upon the *GF* $[p^n]$ and its included *GF* $[p^1]$. Generalizing, we find the *SGS* $[p^{n'}, 2, p^{nn'}]$ (g) as similarly dependent upon the *GF* $[p^{nn'}]$ and its included *GF* $[p^{n'}]$.

This *SGS* $[p^{n'}, 2, p^{nn'}]$ has a transitive group on the $p^{nn'}$ letters which contains a cyclic substitution leaving (any) one letter fixed and permuting in one cycle all the remaining $p^{nn'}-1$ letters. One such cyclic substitution is determined by any primitive root γ of the $GF\,[p^{nn'}]$. The $SGS\,[p^{n'},\ 2,\ p^{nn'}]$ contains $(p^{nn'}-1)/(p^{n'}-1)$ sets s_1. The first $(p^{nn'}-1)/(p^{n'}-1)$ applications of the cyclic substitution connected with γ permute these sets s_1 cyclically in a certain order.

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*My colleague, Mr. Bolza, first used this group-theoretic point of view in obtaining the arrangement (f) for $p = 2$.

†For references to the Galois-field literature, see my paper cited at close of I.

In §4 of my paper referred to above (at close of I), I have tabulated the $SGS\,[p,\ 2,\ p^n]$, for the cases $p^n = 2^2,\ 3^2,$* $5^2,\ 7^2,\ 11^2;\ 2^3,\ 3^3,\ 5^3,\ 7^3;\ 2^4,\ 3^4,\ 5^4;\ 2^5;\ 2^6$, as the defining element of the corresponding linear configuration $LCf\,[p^n]$.

As examples of the $SGS\,[p^{n'},\ 2,\ p^{nn'}]$ with $n' > 1$, I give now two with $n' = 2$:

$SGS\,[4,\ 2,\ 16]$. The $GF\,[2^4]$ has a primitive root γ where $\gamma^4 = 1+\gamma$; the $2^4 = 16$ marks are $0 = *$, $\gamma^i\ (i = 0\ \ldots.\ 14)$. We denote the letters of the $SGS\,[4,\ 2,\ 16]$ by the 16 indices $*\ 01\ \ldots.\ 14$. One set s_1 of four 4-ads is

[* 0 5 10][1 2 4 8][6 7 9 13][11 12 14 3],

from which the five sets s_1 come by use of the cyclic substitution $(*)(01\ \ldots.\ 14)$.

$SGS\,[4,\ 2,\ 64]$. The $GF\,[2^6]$ has the primitive root γ where $\gamma^6 = 1+\gamma+\gamma^3+\gamma^4$. We use the index notation $*\ 01\ \ldots.\ 62$. The twenty-one sets s_1 come by the cyclic substitution $(*)(01\ \ldots.\ 62)$ from the following set of sixteen 4-ads:

[* 0 21 42]
[1 25 56 58][2 50 49 53][3 13 20 57][6 26 40 51][12 52 17 39]
[22 46 14 16][23 8 7 11][24 34 41 15][27 47 61 9][33 10 38 60]
[43 4 35 37][44 29 28 32][45 55 62 36][48 5 19 30][54 31 59 18].

(h). $SGS\,[4,\ 2,\ m = 3p+1 = 12\nu+4]$, where p is any prime of the form $p = 4\nu+1$. This arrangement is given below (III 6).

(i). A $SGS\,[k,\ k,\ m]$ has the element of interest that in it every k-ad of the m letters appears. We have seen above (f) an example: $SGS\,[2,\ 2,\ 2^n]$. Mr. Sylvester gives† a $SGS\,[3,\ 3,\ 9]$, and states that he has a general $SGS\,[3,\ 3,\ 3^n]$.

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* For the $SGS\,[3, 2, 9]$ (e) if we write the symbol thus:

$$\begin{matrix} * & 1 & 5 \\ 0 & 7 & 2 \\ 4 & 6 & 3 \end{matrix}$$

then one cyclic substitution is (*)(01234567).

† Sylvester, "Note on a Nine School-girls Problem" (Messenger of Mathematics, vol. 22, pp. 159-160, p. 192, 1893).

11. *Examples of* $S\{k, l, m\}$ *and* $SGS\{k, l, m\}$.

(a). $S\{m, l, m\}$. For every *exactly* l-ply transitive substitution-group $G^m_{m_l}$ of degree m and of order m_l we have an interesting $S\{m, l, m\}$ consisting of the m_l m-ids derived from any initial m-id by the substitutions of the $G^m_{m_l}$. Mr. Jordan[*] has studied the totality of such groups $G^m_{m_l}$, and separated them into the following systems:

(α). $m = m$, $l = m - 1$ or m. The symmetric $G^m_{m!}$.
(β). $m = m$, $l = m - 2$. The alternating $G^m_{\frac{1}{2}m!}$.
(γ). $m = p^n + 1$, $l = 3$. } p is any prime; n is any integer.
(δ). $m = p^n$, $l = 2$. } Mathieu[†] has given such groups.
(ε). $m = 12$, $l = 5$. Mathieu's[‡] $G^{12}_{12_5}$.
(ζ). $m = 11$, $l = 4$. Mathieu's[‡] $G^{11}_{11_4}$.
(η). $m = m$, $l = 1$. The *regular forms* G^m_m of all group G_m of order m.

(b). $SGS\{m, l, m\}$. If the $G^m_{m_l}$ of (a) has a succession of sub-groups $G^m_{m_g}$ $(g = 1, \ldots . l)$, the $G^m_{m_g}$ being (for every g) exactly g-ply transitive and a sub-group of the $G^m_{m_{g+1}}$, then the $G^m_{m_l}$ may be exhibited as a rectangular array, first with respect to the G^m_m, and then the lines of this array as a rectangular array with respect to the $G^m_{m_2}$, and so on. Correspondingly the $S\{m, l, m\}$ is exhibited as a $SGS\{m, l, m\}$.

But do any of the groups $G^m_{m_l}$ (a) contain such a series of sub-groups $G^m_{m_g}$?

(aη). This case is trivial. Every $S\{k, 1, m\}$ is a $SGS\{k, 1, m\}$.

(aζ). The $G^{12}_{12_5}$ has no such series, since there is no $G^{12}_{12_4}$.

(aε). The $G^{11}_{11_4}$ has no such series, since there is no $G^{11}_{11_3}$.

(aδ). Every $G^{p^n}_{(p^n)_2}$ has such a series, since it contains[§] a (self-conjugate) transitive $G^{p^n}_{p^n}$.

(aγ). A $G^m_{m_3}$ $(m = p^n + 1)$ contains a $G^m_{m_2}$ if at all only when

(aγ)$_1$. $m = p^n + 1 = 3$, $m - 1 = 2$. The $G^3_{3_3}$ is the symmetric $G^3_{3!}$; see (aα)$_1$.

(aγ)$_2$. $m = p^n + 1 = 4$, $m - 2 = 2$. The $G^4_{4_3}$ is the symmetric $G^4_{4!}$; see (aα)$_2$.

(aγ)$_3$. $m = p^n + 1 = p'^{n'}$. If the $G^m_{m_3}$ does contain a doubly-transitive $G^m_{m_2}$, then the latter will contain a simply transitive $G^m_{m_1}$ (aδ), and so the $G^m_{m_3}$ has the series $G^m_{m_3}$, $G^m_{m_2}$, $G^m_{m_1}$. The type (a$\gamma)_3$ contains the types (a$\gamma)_1$ and (a$\gamma)_2$.

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[*] Jordan, "Recherches sur les substitutions" (Journal de Mathématiques, ser. 2, vol. 17, pp. 351–367, 1872).

[†] Mathieu, "Mémoire sur le nombre de valeurs . . . " (ibid., ser. 2, vol. 5, pp. 9–42, 1860), p. 38. In §15 (1) I exhibit Mathieu's group (δ).

[‡] Mathieu, "Mémoire sur l'étude des fonctions de plusieurs quantités, . . . " (ibid., ser. 2, vol. 6, pp. 241–323, 1861), p. 270.

[§] Jordan, loc. cit., p. 356, §10.

(aβ). The alternating $G^m_{m_{m-2}}$ $(m \geqq 4)$ contains a $G^m_{m_{m-3}}$ if at all only when

(a$\beta)_1$. $m - 3 = 1$; $m = 4$. The $G^4_{4_2}$ is the alternating G^4_{12} (aβ) and likewise the only $G^{2^2}_{(2^2)_2}$ (aδ); it contains the (self-conjugate) transitive four-group G^4_4, and so has the series $G^4_{4_2}$, $G^4_{4_1}$.

(a$\beta)_2$. $m - 3 = 2$; $m = p^n = 5$. The alternating $G^5_{5_3}$ is the icosahedron G_{60}, and contains no sub-group G_{20}. The $G^5_{5_3}$ has then no series $G^5_{5_3}$, $G^5_{5_2}$, $G^5_{5_1}$. (The $G^5_{5_2}$ (aδ) is the metacyclic G^5_{20}; half its substitutions are odd.)

(a$\beta)_3$. $m - 3 = 3$; $m = p^n + 1 = 6$. The alternating $G^6_{6_4}$ contains no transitive sub-group G^6_6, and hence no series $G^6_{6_4}$, $G^6_{6_3}$, $G^6_{6_2}$, $G^6_{6_1}$. (The $G^6_{6_3}$ (aγ) has half its substitutions odd.)

(aα). The symmetric $G^m_{m_{m-1}}$ $(m \geqq 3)$ contains always the alternating $G^m_{m_{m-2}}$, and contains a series $G^m_{m_g}$ $(g = 1 \ldots . l)$ in and only in the two cases

(a$\alpha)_1$. $m = 3$. The symmetric $G^3_{3_2}$ contains the cyclic G^3_3.

(a$\alpha)_2$. $m = 4$. The symmetric $G^4_{4_3}$ contains the alternating $G^4_{4_2}$ and this the four-group G^4_4.

Thus the only $G^m_{m_l}$ $(l > 1)$ containing a series of $G^m_{m_g}$ $(g = 1 \ldots . l)$ are the symmetric $G^4_{4_3}$, $G^3_{3_2}$ and the alternating $G^4_{4_2}$, and all doubly-transitive $G^m_{m_2}$ $(m = p^n)$, and certain exceptional (a$\gamma)_3$ triply-transitive $G^m_{m_3}$ $(m = p^n + 1 = p'^{n'})$. Each of these yields a $SGS\{m, l, m\}$. The $G^3_{3_2}$, $G^4_{4_2}$, $G^4_{4_3}$ yield (c) (d) (e).

(c). $SGS\{3, 2, 3\} \equiv SGS\{3, 3, 3\}$. $\{abc\}\{bca\}\{cab\}$, $\{acb\}\{bac\}\{cba\}$. There is but one $SGS\{3, 2, 3\}$.

(d). $SGS\{4, 2, 4\}$.

$$\{abcd\}\{badc\}\{cdab\}\{dcba\},$$
$$\{acdb\}\{bdca\}\{cabd\}\{dbac\},$$
$$\{adbc\}\{bcad\}\{cbda\}\{dacb\}.$$

(e). $SGS\{4, 3, 4\} \equiv SGS\{4, 4, 4\}$. To the $SGS\{4, 2, 4\}$ of (d) adjoin the $SGS\{4, 2, 4\}$ obtained therefrom by the substitution (cd).

12. $S[\underset{\cdot}{k}, l, m]$, $S\{\underset{\cdot}{k}, l, m\}$ and $SGS\{\underset{\cdot}{k}, l, m\}$; $(k \geqq l)$. The definition of one of these $\underset{\cdot}{k}$-adic ($\underset{\cdot}{k}$-idic) systems differs from the definition of its corresponding k-adic (k-idic) system only by the substitution throughout for the term g-ad (g-id) of the term $\underset{\cdot}{g}$-ad ($\underset{\cdot}{g}$-id). *It is important to note that the remark* $m \geqq k$ *is now removed*, since in a $\underset{\cdot}{k}$-ad ($\underset{\cdot}{k}$-id) repetitions are allowed. Since only for $g = 1$

is *every* $\underset{\cdot}{g}$-ad ($\underset{\cdot}{g}$-id) likewise a g-ad (g-id), one of these $\underset{\cdot}{k}$-adic ($\underset{\cdot}{k}$-idic) systems may perhaps coincide with its corresponding k-adic (k-idic) system only in the trivial case $l = 1$. (It is easy to see that the $SGS[\underset{\cdot}{k}, l, m]$ with $l > 1$ is non-existent, while for $l = 1$ it is merely a $S[\underset{\cdot}{k}, 1, m]$.)

A $S[\underset{\cdot}{k}, k, m]$ ($S\{\underset{\cdot}{k}, k, m\}$) is simply the totality of $\underset{\cdot}{k}$-ads ($\underset{\cdot}{k}$-ids) of the m letters, and contains the $S[k, k, m]$ ($S\{k, k, m\}$).

13. *Theorems concerning* $S[\underset{\cdot}{k}, l, m]$, $S\{\underset{\cdot}{k}, l, m\}$, $SGS\{\underset{\cdot}{k}, l, m\}$.

(a). In m letters there are in all m^l $\underset{\cdot}{l}$-ids. On the face of every $\underset{\cdot}{k}$-id there are k_l l-idic positions. The $S\{\underset{\cdot}{k}, l, m\}$ contains m^l $\underset{\cdot}{k}$-ids.

(b). Every set s_g of a $SGS\{\underset{\cdot}{k}, l, m\}$ is a $SGS\{\underset{\cdot}{k}, g, m\}$ and contains m^g $\underset{\cdot}{k}$-ids. Every set s_h ($h > g$) contains m^{h-g} sets s_g. The $SGS\{\underset{\cdot}{k}, l, m\}$ contains m^{l-g} sets s_g. Every $SGS\{\underset{\cdot}{k}, l, m\}$ is thus a configuration.

(c). If in a $S[\underset{\cdot}{k}, l, m]$ ($S\{\underset{\cdot}{k}, l, m\}$) we suppress all $\underset{\cdot}{k}$-ads ($\underset{\cdot}{k}$-ids) not containing a certain $\underset{\cdot}{g}$-ad (g-id in a certain position) ($g < l$), and then in the remaining $\underset{\cdot}{k}$-ads ($\underset{\cdot}{k}$-ids) suppress that $\underset{\cdot}{g}$-ad ($\underset{\cdot}{g}$-id), we have a $S[k \underset{\cdot}{-} g, l - g, m]$ ($S\{k \underset{\cdot}{-} g, l - g, m\}$).

(d). If in (c) the $S\{\underset{\cdot}{k}, l, m\}$ is a $SGS\{\underset{\cdot}{k}, l, m\}$, then the resulting $S\{k \underset{\cdot}{-} g, l - g, m\}$ is a $SGS\{k \underset{\cdot}{-} g, l - g, m\}$.

(e). From (c) (d) arise (as in §§4, 5, 8, 9) the notions: *maximum* $S[\underset{\cdot}{k}, l, m]$, $S\{\underset{\cdot}{k}, l, m\}$, $SGS\{\underset{\cdot}{k}, l, m\}$.

(f). If in a $S\{\underset{\cdot}{k}, l, m\}$ ($SGS\{\underset{\cdot}{k}, l, m\}$) we suppress in every $\underset{\cdot}{k}$-id the $\underset{\cdot}{g}$-id ($g \leqq k - l$) occupying a certain g-idic position on the face of the $\underset{\cdot}{k}$-id, we have left a $S\{k \underset{\cdot}{-} g, l, m\}$ ($SGS\{k \underset{\cdot}{-} g, l, m\}$).

(g). The process (f) gives us, if $k \geqq 2l$, $g \geqq l$, $\therefore k - g \geqq l$, two systems $S\{k \underset{\cdot}{-} g, l, m\}$, $S\{\underset{\cdot}{g}, l, m\}$, ($SGS\{k \underset{\cdot}{-} g, l, m\}$, $SGS\{\underset{\cdot}{k}, l, m\}$) with an interesting 1—1 correspondence between the $(k \underset{\cdot}{-} g)$-ids of one and the $\underset{\cdot}{g}$-ids of the other.

(h). The process (f) gives us, if $g < l$ and $g \leqq k - l$, a system $S\{k \underset{\cdot}{-} g, l, m\}$ ($SGS\{k \underset{\cdot}{-} g, l, m\}$) containing (c, d) $m^g S\{k \underset{\cdot}{-} g, l - g, m\}$ ($SGS\{k \underset{\cdot}{-} g, l - g, m\}$).

(i). m $S\{\underset{\cdot}{k}, l, m\}$ ($SGS\{\underset{\cdot}{k}, l, m\}$, no two of whose $\underset{\cdot}{k}$-ids have an $(l + 1)$-id in common, taken together constitute a $S\{\underset{\cdot}{k}, l + 1, m\}$ ($SGS\{\underset{\cdot}{k}, l + 1, m\}$).

14. *Examples of* $S[\underset{\cdot}{k}, l, m]$.

(a). $S[\underset{\cdot}{k}, k - 1, m]$ (k, m any integers). The k letters $a_{i_1} a_{i_2} \dots a_{i_k}$ constitute a $\underset{\cdot}{k}$-ad when and only when

$$i_1 + i_2 + \dots + i_k \equiv 0 \pmod{m}.$$

(b).[*] $S[\underset{\cdot}{k}, 2, m]$. If to the k-ads of a $S[k, 2, m]$ we adjoin the m $\underset{\cdot}{k}$-adic sequences $[a_i a_i \dots a_i]$ $(i = 1, \dots m)$ we have a $S[\underset{\cdot}{k}, 2, , m]$.

(c). $S[\underset{\cdot}{3}, 2, m]$. Complete table of all classes for $m = 1, 2, 3, 4$:

$S[\underset{\cdot}{3}, 2, 1]$. (1) $[aaa]$.
$S[\underset{\cdot}{3}, 2, 2]$. (1) $[aaa]$ $[abb]$.
$S[\underset{\cdot}{3}, 2, 3]$. (1) $[aaa]$ $[abc]$ $[bbb]$ $[ccc]$.
(2) $[aab]$ $[acc]$ $[bbc]$.
$S[\underset{\cdot}{3}, 2, 4]$. (1) $[aaa]$ $[abb]$ $[acc]$ $[add]$ $[bcd]$.
(2) $[aaa]$ $[abb]$ $[acd]$ $[bcc]$ $[bdd]$.

The case $S[\underset{\cdot}{3}, 2, 3]$ illustrates the fact that in a $S[\underset{\cdot}{k}, l, m]$ the number of $\underset{\cdot}{k}$-ads depends not only on the characters (k, l, m) but also on the internal structure of the $S[\underset{\cdot}{k}, l, m]$.

15. *Examples of* $S\{\underset{\cdot}{k}, l, m\}$ *and* $SGS\{\underset{\cdot}{k}, l, m\}$.

(a). $SGS\{\underset{\cdot}{2}, 2, 2\}$. $\{aa\}\{bb\}$, $\{ab\}\{ba\}$. There is no other $SGS\{\underset{\cdot}{2}, 2, 2\}$.

(b). $S\{\underset{\cdot}{3}, 2, 2\}$. $\{aaa\}$ $\{abb\}$ $\{bab\}$ $\{bba\}$. There is but one class of $S\{\underset{\cdot}{3}, 2, 2\}$. There is no $SGS\{\underset{\cdot}{3}, 2, 2\}$, and hence no $SGS\{\underset{\cdot}{k}, l, 2\}$ $(k \geqq l \geqq 2)$ except the $SGS\{\underset{\cdot}{2}, 2, 2\}$ (a).

(c). $S\{\underset{\cdot}{k}, 2, m\}$ $(SGS\{\underset{\cdot}{k}, 2, m\})$. Every $S\{k, 2, m\}$ $(SGS\{k, 2, m\})$ may be at once extended to a $S\{\underset{\cdot}{k}, 2, m\}$ $(SGS\{\underset{\cdot}{k}, 2, m\})$ by adjoining the m $\underset{\cdot}{k}$-idic sequences $\{a_i a_i \dots a_i\}$ $(i = 1, \dots m)$. (These form the m^{th} $SGS\{\underset{\cdot}{k}, 1, m\}$ of the $SGS\{\underset{\cdot}{k}, 2, m\}$).

(c′). Whenever a $S\{\underset{\cdot}{k}, l, m\}$ $(SGS\{\underset{\cdot}{k}, l, m\})$ contains a $S\{k, l, m\}$ $(SGS\{k, l, m\})$, no $\underset{\cdot}{k}$-id containing a repeated letter contains more than $l - 1$ distinct letters.

(d). $SGS\{\underset{\cdot}{2}, 2, 3\}$. There are exactly two $SGS\{\underset{\cdot}{2}, 2, 3\}$; they are not equivalent. The matricular arrangement

$$\begin{matrix} \{aa\} & \{bb\} & \{cc\} \\ \{bc\} & \{ca\} & \{ab\} \\ \{cb\} & \{ac\} & \{ba\} \end{matrix}$$

read by rows gives one system, and by columns the other system.

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[*] This $S[\underset{\cdot}{k}, 2, m]$ for $k = 3$ was of much use in my construction of triple systems $S[3, 2, m]$ for all permissible values of m, $m = 6\mu' + 1, 6\mu' + 3$: "Concerning Triple Systems" (Mathematische Annalen, vol. 43, pp. 271-285, 1893).

(e). $SGS\{\underset{\bullet}{3}, 2, 3\}$. The matricular arrangement (d), which exhibits the nine $\underset{\bullet}{2}$-ids in two ways as a $SGS\{\underset{\bullet}{2}, 2, 3\}$, may at once be extended to a $SGS\{\underset{\bullet}{3}, 2, 3\}$ by affixing a letter to every $\underset{\bullet}{2}$-id, the $\underset{\bullet}{2}$-ids of the same column (or row) receiving the same letter, the three columns (or rows) receiving the three letters a, b, c in any order. The affixed letter is to occupy any the same position in every resulting $\underset{\bullet}{3}$-id.

(e′). $SGS\{\underset{\bullet}{k}, l, m\}$. Every $SGS\{\underset{\bullet}{k}, l, m\}$ $(k > l > 1)$ may be derived in the way indicated in (e) from a matricular arrangement exhibiting $m^2\,SGS\{k \underset{\bullet}{-} 1, l - 2, m\}$—(where in case $l = 2$ a $SGS\{k \underset{\bullet}{-} 1, 0, m\}$ means a $(k \underset{\bullet}{-} 1)$-id)—in two ways as a $SGS\{k \underset{\bullet}{-} 1, l, m\}$. (Compare §13 f.)

(f). $SGS\{\underset{\bullet}{3}, 2, 3\}$ with the property that every set $s_1 \equiv SGS\{\underset{\bullet}{3}, 1, 3\}$ is invariant under the substitution (abc). By (e′, d) every such $SGS\{\underset{\bullet}{3}, 2, 3\}$ arises by *prefixing* the letters a, b, c or b, c, a or c, a, b to the $\underset{\bullet}{2}$-ids of the respective columns of the arrangement (d). There are then three such $SGS\{\underset{\bullet}{3}, 2, 3\}$. The first forms a class by itself; the last two are equivalent.

(g). $SGS\{\underset{\bullet}{3}, 3, 3\}$. The three $SGS\{\underset{\bullet}{3}, 2, 3\}$ of (f) constitute a $SGS\{\underset{\bullet}{3}, 3, 3\}$.

(h). $SGS\{\underset{\bullet}{4}, 2, 4\}$. The $SGS\{4, 2, 4\}$ of §11 d whose sets $s_1 \equiv SGS\{4, 1, 4\}$ are separately invariant under the four-group G_4^4, yields a similar $SGS\{\underset{\bullet}{4}, 2, 4\}$ by adjoining as fourth set s_1 the four sequences $\{aaaa\}$ . . $\{dddd\}$ (c).

(i). Two-fold $SGS\{\underset{\bullet}{3}, 2, 4\}$. We seek ultimately (k) a $SGS\{\underset{\bullet}{4}, 4, 4\}$ whose sets $s_1 \equiv SGS\{\underset{\bullet}{4}, 1, 4\}$ are separately invariant under the G_4^4, and, from the $SGS\{\underset{\bullet}{4}, 2, 4\}$ (h) by dropping in every $\underset{\bullet}{4}$-id the first position by the process (e′), obtain the matricular arrangement—two-fold $SGS\{\underset{\bullet}{3}, 2, 4\}$—

$$\begin{array}{llll}
\{aaa\} & \{bbb\} & \{ccc\} & \{ddd\} \\
\{bcd\} & \{adc\} & \{dab\} & \{cba\} \\
\{cdb\} & \{dca\} & \{abd\} & \{bac\} \\
\{dbc\} & \{cad\} & \{bda\} & \{acb\}.
\end{array}$$

(j). Two-fold $SGS\{\underset{\bullet}{2}, 2, 4\}$. By dropping any position in the $\underset{\bullet}{3}$-ids of the two-fold $SGS\{\underset{\bullet}{3}, 2, 4\}$ (i) we obtain a two-fold $SGS\{\underset{\bullet}{2}, 2, 4\}$. The three arrangements so obtained are identical, except as to arrangement of the rows.

(i). The twelve positional-columns of the $SGS\{\underset{\bullet}{3}, 2, 4\}$ (i) are derived from the first by the G_{12}^4. The three positional-columns of the first column of 3-ids are by pairs conjugate under the G_{12}^4 but not under the G_4^4. Any two ordered positional-columns, conjugate under the G^4_{12} but not under the G^4_4, determine a column of $\underset{\cdot}{2}$-ids differing only in literal notation and in order from the first column of the arrangement (j), and (since the G^4_4 is self-conjugate under the G^4_{12}), this column of $\underset{\cdot}{2}$-ids determines a two-fold $SGS\{\underset{\cdot}{2}, 2, 4\}$. Hence *any* three ordered positional-columns by pairs conjugate under the G^4_{12} but not under the G^4_4, determine a column of $\underset{\cdot}{3}$-ids which by the G^4_4 leads to a two-fold $SGS\{\underset{\cdot}{3}, 2, 4\}$.

From the two-fold $SGS\{\underset{\cdot}{3}, 2, 4\}$ (i) we obtain, by applying the substitutions of the G^4_4 simultaneously and independently to the several positional-columns, in all sixteen two-fold $SGS\{\underset{\cdot}{3}, 2, 4\}$. (We may leave say the first-position letters unchanged.) Any row is derived from its initial $\underset{\cdot}{3}$-id by the G^4_4. Any row is sufficiently specified by its initial $\underset{\cdot}{3}$-id, and any two-fold $SGS\{\underset{\cdot}{3}, 2, 4\}$ arrangement by its initial column of $\underset{\cdot}{3}$-ids. Obviously every $\underset{\cdot}{3}$-id in the four letters appears once in every line and once in every column somewhere in these sixteen arrangements. By prefixing an a, b, c, d to every $\underset{\cdot}{3}$-id of the first, second, third, fourth column respectively, we have then the 4^4 $\underset{\cdot}{4}$-ids of the four letters arranged by rows in 4^3 sets $s_1 \equiv SGS\{\underset{\cdot}{4}, 1, 4\}$ (separately invariant under the G^4_4), and these by fours in 4^2 sets $s_2 \equiv SGS\{\underset{\cdot}{4}, 2, 4\}$ (by the theory of (e′)). Denoting these 4^2 $SGS\{\underset{\cdot}{4}, 2, 4\}$ by their respective first columns, I group them by fours in 4 sets $s_3 \equiv SGS\{\underset{\cdot}{4}, 3, 4\}$ and have thus a

(k). $SGS\{\underset{\cdot}{4}, 4, 4\}$ with sets $s_1 \equiv SGS\{\underset{\cdot}{4}, 1, 4\}$ separately invariant under the G^4_4:

{*aaaa*}	{*aabb*}	{*aacc*}	{*aadd*}
{*abcd*}	{*abdc*}	{*abab*}	{*abba*}
{*acdb*}	{*acca*}	{*acbd*}	{*acac*}
{*adbc*}	{*adad*}	{*adda*}	{*adcb*};

{*abbb*}	{*abaa*}	{*abdd*}	{*abcc*}
{*aadc*}	{*aacd*}	{*aaba*}	{*aaab*}
{*adca*}	{*addb*}	{*adac*}	{*adbd*}
{*acad*}	{*acbc*}	{*accb*}	{*acda*};

{*accc*}	{*acdd*}	{*acaa*}	{*acbb*}
{*adab*}	{*adba*}	{*adcd*}	{*addc*}
{*aabd*}	{*aaac*}	{*aadb*}	{*aaca*}
{*abda*}	{*abcb*}	{*abbc*}	{*abad*};

$$\begin{array}{c|c|c|c}
\{addd\} & \{adcc\} & \{adbb\} & \{adaa\} \\
\{acba\} & \{acab\} & \{acdc\} & \{accd\} \\
\{abac\} & \{abbd\} & \{abca\} & \{abdb\} \\
\{aacb\} & \{aada\} & \{aaad\} & \{aabc\}.
\end{array}$$

It will be noticed that the four columns of each row come out of the first one by simultaneous application to the third- and fourth-positional columns of the (same) substitutions of the G^4_4, and the four rows come out of the first one by simultaneous application to the second-, third-, and fourth-positional columns of the (same) substitutions of the G^4_4.

That every row above is a $SGS\{\underset{\cdot}{4}, 3, 4\}$ follows (first) for a $\underset{\cdot}{3}$-id, one of whose letters is in the first position on the $\underset{\cdot}{4}$-id in accordance with the theory of (e′) and the remarks above, and (second) for a $\underset{\cdot}{3}$-id occupying the positions second, third, and fourth on the $\underset{\cdot}{4}$-id, because, when the first-position letters are everywhere suppressed, the four rows are identical (e. g. the $\{aaa\}$ comes in the $s_1 \equiv SGS\{\underset{\cdot}{4}, 1, 4\}$ containing the $\{aaaa\}$, $\{abbb\}$, $\{accc\}$, $\{addd\}$, and so each constitutes a $SGS\{\underset{\cdot}{3}, 3, 4\}$.

(l). Two-fold $SGS\{m \underset{\cdot}{-} 1, 2, m\}$ ($m = p^n$, p any prime; $m > 2$). This arrangement is a direct generalization of the two-fold $SGS\{\underset{\cdot}{3}, 2, 4\}$ (i). We use Mathieu's exactly doubly transitive $G^m_{m_2}$ (§11 aδ). The $m = p^n$ letters are the p^n marks ξ of the Galois-field $GF[p^n]$. The substitutions of the $G^m_{m_2}$ given analytically are

$$\xi' = \alpha\xi + \beta, \qquad (\alpha \neq 0)$$

where β is any mark and α any mark except 0 of the $GF[p^n]$. The G^m_m,

$$\xi' = \xi + \beta, \qquad (\beta \text{ any mark})$$

is self-conjugate in the $G^m_{m_2}$.

From any initial m-id $\{\xi_1, \xi_2 \ldots\ldots \xi_m\}$ we obtain by the $G^m_{m_2}$ in all $m_2 = m(m-1)$ m-ids; these are conjugate under the $G^m_{m_2}$. They are in sets of m conjugate under the G^m_m; the m substitutions $\xi' = \alpha\xi + \beta$, with α fixed and β arbitrary, give such a set of m m-ids.

The two-fold $SGS\{m \underset{\cdot}{-} 1, 2, m\}$ to be constructed is a matricular arrangement $\{s_{ij}\}$ ($i, j = 1 \ldots\ldots m$) whose elements s_{ij} are $(m \underset{\cdot}{-} 1)$-ids,

$$s_{ij} = \{\xi_{ij1} \ldots\ldots \xi_{ijk} \ldots\ldots \xi_{ij\,m-1}\}.$$

We take any $(m-1)$-id $\{\alpha_1 \ldots\ldots \alpha_{m-1}\}$ of the $m-1$ marks $\xi \neq 0$, any $(m \underset{\cdot}{-} 1)$-id

$\{\beta_1 \ldots . \beta_{m-1}\}$, and any two m-ids $\{\gamma_1 \ldots . \gamma_m\}$, $\{\xi_1 \ldots . \xi_m\}$ of the m marks ξ, and set

$$\xi_{ijk} = \alpha_k \xi_i + \beta_k + \gamma_j. \qquad \begin{pmatrix} i, j = 1 \ldots . m \\ k \;\; = 1 \ldots . m - 1 \end{pmatrix}$$

That this arrangement $\{s_{ij}\}$ is in fact a two-fold $SGS\{m - 1, 2, m\}$ may be easily verified analytically.

(m). $SGS\{m, 2, m\}$ $(m = p^n)$. From the arrangement (l) by the process used in (e), we obtain a $SGS\{m, 2, m\}$. [For $m = 2$ see (a)].

(n). Two-fold $SGS\{k, 2, m\}$ $(2 \leqq k \leqq m - 1,\ m = p^n)$. We apply the process of §13 f to the arrangement (l).

(o). $SGS\{p, p, p\}$ (p any prime); $SGS\{k, l, p\}$ $(p \geqq k \geqq l)$. This $SGS\{p, p, p\}$ is connected with the Abelian or commutative group G_{p^p} determined by p generators $A_1 \ldots . A_p$, each of period p and a certain succession of sub-groups G_{p^g} $(g = 1, 2 \ldots . p - 1)$ having as *first* property this, that the G_{p^g} lies in the $G_{p^{g+1}}$ $(g = 1 \ldots . p - 2)$. Now we arrange the p^p elements of the G_{p^p} with respect to the G_{p^1} by p's in p^{p-1} sets s_1, and these p^{p-1} sets s_1 with respect to the G_{p^2} by p's in p^{p-2} sets s_2, $\ldots .$ and these p^{p-g} sets s_g with respect to the $G_{p^{g+1}}$ by p's in p^{p-g-1} sets s_{g+1}, $\ldots .$ and these p^2 sets s_{p-2} with respect to the $G_{p^{p-1}}$ by p's in p sets s_{p-1}; the individual elements play the rôle of p^p sets s_0, and the totality of p^p elements plays the rôle of the one set s_p.

We define the succession of sub-groups G_{p^g} $(g = 1 \ldots . p - 1)$ by a succession of $p - 1$ independent generators B_g $(g = 1 \ldots . p - 1)$,

$$B_g = A_1^{e_{g1}} \ldots . A^{e_{gi}} \ldots . A_p^{e_{gp}};$$

the G_{p^h} has the generators $B_1 \ldots . B_h$.

We work with p letters $a_0 a_1 \ldots . a_{p-1}$ or say a_f where the f's are integers taken modulo p. We associate with every element $A_1^{e_1} \ldots . A_i^{e_i} \ldots . A_p^{e_p} = A_{e_1 \ldots e_i \ldots e_p}$ of the G_{p^p} a corresponding p-id $\{a_{e_1} \ldots . a_{e_i} \ldots . a_{e_p}\}$ or say $\{e_1 \ldots . e_i \ldots . e_p\}$ or say $\{\{e_i\}\}$ in the p letters. The arrangement of the elements leads to this arrangement of the p-ids:

Any particular p-id $\{\{e_i'\}\}$ lies in and determines the set s_h of p^h p-ids $\{\{e_i\}\}$

$$e_i = e_i' + \sum_{g=1}^{g=h} m_g e_{gi}, \qquad (i = 1 \ldots . p)$$

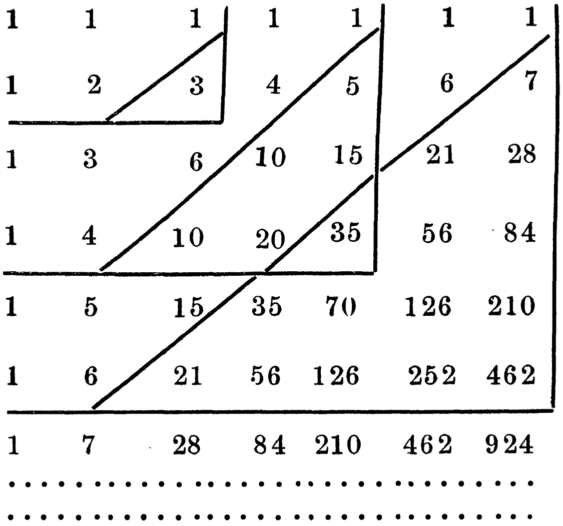
where the $m_1 \ldots . m_h$ run independently over the integers 0, 1, $\ldots .$ $p - 1$.

Now such a set s_h is a $S\{p, h, p\}$ if it contains in its p^h p-ids $\{\{e_i\}\}$ every h-id of the p letters once and only once in every position, that is, analytically, if the rectangular array

$$E_h \equiv (e_{gi}) \qquad \begin{pmatrix} g = 1 \ldots . h \\ i = 1 \ldots . p \end{pmatrix}$$

has no determinant of order h which vanishes. If the $p - 1$ arrays E_h $(h = 1 \ldots . p - 1)$ have every one this property, then our arrangement of p^p p-ids in p letters is a $SGS\{p, p, p\}$. This is the *second* property which the succession of $p - 1$ sub-groups G_{p^g} $(g = 1 \ldots . p - 1)$ must possess (not indeed absolutely in the G_{p^p}, but) relatively to the p chosen generators $A_1 \ldots . A_p$ of the G_{p^p}.

The Pascal arithmetical triangle when taken in the form



1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462
1	7	28	84	210	462	924

. . . .

. . . .

furnishes us in an interesting way with an effective rectangular array $(e_{gi}) \begin{pmatrix} g = 1 \ldots . p - 1 \\ i = 1 \ldots . p \end{pmatrix}$ *for every prime* p. If we denote the elements of the triangle by u_{gi} $(g, i = 1, 2, 3 \ldots .)$, we take simply

$$e_{gi} \equiv u_{gi} \ (\text{mod. } p). \qquad \begin{pmatrix} g = 1 \ldots . p - 1 \\ i = 1 \ldots . p \end{pmatrix}$$

We have
$$u_{gi} = \frac{(g + i - 2)!}{(g - 1)!\,(i - 1)!} = \frac{(g + i - 2)_{g-1}}{(g - 1)!}, \qquad (u_{g1} = u_{1i} = 1)$$

so that $e_{gi} \equiv u_{gi} \equiv 0$ (mod. p) whenever $g + i - 2 \geqq p$. [The arrays (e_{gi}) for $p = 3, 5, 7$ are indicated in the triangle.] The array is indeed effective: the determinant $|u_{gi}| \left(\begin{smallmatrix} g = 1 \dots\dots h \\ i = i_1 \dots\dots i_h \,;\, i_f \leqq p \end{smallmatrix}\right)$ of any order $h\,(\leqq p-1)$ lying in the first h rows and in any h of the first p columns is evaluated* by the formula

$$\prod_{g=2}^{g=h}(g-1)\,!\,.\,|u_{gi}| = |i^{g-1}| = \prod_{f_1>f_2}(i_{f_1}-i_{f_2}), \qquad \left(\begin{smallmatrix} g,\, f_1,\, f_2 = 1 \dots\dots h \\ i = i_1 \dots\dots i_h \,;\, i_f \leqq p \end{smallmatrix}\right)$$

so that indeed

$$|e_{gi}| \equiv |u_{gi}| \not\equiv 0 \ (\text{mod.}\ p). \qquad \left(\begin{smallmatrix} g = 1 \dots\dots h \\ i = i_1 \dots\dots i_h \,;\, i_f \leqq p \end{smallmatrix}\right)$$

Instead of making use of the Pascal triangle u_{gi} $(g,\, i = 1,\, 2,\, 3 \dots\dots)$ we might use the system v_{gi} $(g,\, i = 1,\, 2 \dots\dots)$,

$$v_{gi} = i^{g-1}, \quad v_{1i} = 1\,.$$

The triangle however has certain advantages.

From every $SGS(\underset{\cdot}{p},\, p,\, p)$ by the definition we have (many) $SGS\{\underset{\cdot}{p},\, l,\, p\}$ $(l \leqq p)$, and from every $SGS\{\underset{\cdot}{p},\, l,\, p\}$, by the process of §13f (many) $SGS\{\underset{\cdot}{k},\, l,\, p\}$ $(l \leqq k \leqq p)$.

(p). $SGS\{\underset{\cdot}{m},\, m,\, m\}$ $(m = p^n$, p any prime, n any integer); $SGS\{\underset{\cdot}{k},\, l,\, m\}$ $(m = p^n \geqq k \geqq l)$. This $SGS\{\underset{\cdot}{p^n},\, p^n,\, p^n\}$ is a direct analytic generalization of the $SGS\{\underset{\cdot}{p},\, p,\, p\}$ (o).

We work with $m = p^n$ letters a_ξ where ξ is a mark of the Galois-field $GF[p^n]$, and have at once the $S\{\underset{\cdot}{m},\, m,\, m\}$ of (all) m^m $\underset{\cdot}{m}$-ids $\{a_{\xi_1} \dots\dots a_{\xi_i} \dots\dots a_{\xi_m}\}$ or say $\{\xi_1 \dots\dots \xi_i \dots\dots \xi_m\}$ or say $\{\{\xi_i\}\}$. We have at disposal a rectangular array

$$E = (\varepsilon_{gi}), \qquad \left(\begin{smallmatrix} g = 1 \dots\dots m-1 \\ i = 1 \dots\dots m \end{smallmatrix}\right)$$

where the ε_{gi} are marks of the $GF[p^n]$ and where no determinant

$$|\varepsilon_{gi}| \qquad \left(\begin{smallmatrix} g = 1 \dots\dots h \\ i = i_1 \dots\dots i_h \,;\, i_f \leqq m \end{smallmatrix}\right)$$

vanishes. Then to exhibit our $S\{\underset{\cdot}{m},\, m,\, m\}$ as a $SGS\{\underset{\cdot}{m},\, m,\, m\}$ we let any particular $\underset{\cdot}{m}$-id $\{\{\xi'_i\}\}$ lie in and determine the set s_h of m^h $\underset{\cdot}{m}$-ids $\{\{\xi_i\}\}$,

$$\xi_i = \xi'_i + \sum_{g=1}^{g=} \mu_g\, \varepsilon_{gi}, \qquad (i = 1 \dots\dots m)$$

where the $\mu_1 \dots\dots \mu_h$ run independently over the m marks of the $GF[m = p^n]$.

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* Scott's Determinants, p. 116, §4; Baltzer's Determinanten, 5th ed., p. 87, §3 I.

An effective array $E = (\varepsilon_{gi})$ is this one:

$$\varepsilon_{gi} = \varepsilon_i^{g-1};\ \varepsilon_{1i} = 1, \qquad \begin{pmatrix} g = 1 \ldots . m-1 \\ i = 1 \ldots , m \end{pmatrix}$$

where $\{\{\varepsilon_i\}\}$ is any m-id of the m marks ξ, for we have

$$|\varepsilon_{gi}| = |\varepsilon_i^{g-1}| = \prod_{f_1 > f_2} \left(\varepsilon_{i_{f_1}} - \varepsilon_{i_{f_2}}\right) \neq 0. \qquad \begin{pmatrix} g, f_1, f_2 = 1 \ldots . h \\ i = i_1 \ldots . i_h;\ i_f \leqq m \end{pmatrix}$$

16. *Composition theorems.*

(a). *Given a $S\{\underset{\cdot}{k}, l, m\}^{(a)}$ in the m letters $a_i\,(i = 1 \ldots . m)$ and a $S\{\underset{\cdot}{k}, l, m\}^{(b)}$ in the n letters $b_j\,(j = 1 \ldots . n)$: to construct a $S\{\underset{\cdot}{k}, l, mn\}^{(c)}$ in the mn letters $c_{ij} \begin{pmatrix} i = 1 \ldots . m \\ j = 1 \ldots . n \end{pmatrix}$.*

Two $\underset{\cdot}{k}$-ids $\{a_{i_1} \ldots . a_{i_f} \ldots . a_{i_k}\}\{b_{j_1} \ldots . b_{j_f} \ldots . b_{j_k}\}$ *compound* uniquely into a $\underset{\cdot}{k}$-id $\{c_{i_1 j_1} \ldots . c_{i_f j_f} \ldots . c_{i_k j_k}\}$. The *composition* of the $S\{\underset{\cdot}{k}, l, m\}^{(a)}$ and the $S\{\underset{\cdot}{k}, l, n\}^{(b)}$ into a $S\{\underset{\cdot}{k}, l, mn\}^{(c)}$ is at once effected by compounding every $\underset{\cdot}{k}$-id of the first system with every $\underset{\cdot}{k}$-id of the second system.

(b). *Given a $SGS\{\underset{\cdot}{k}, l, m\}^{(a)}$ and a $SGS\{\underset{\cdot}{k}, l, n\}^{(b)}$: to construct a $SGS\{\underset{\cdot}{k}, l, mn\}^{(c)}$.*

We compound by (a) the $S\{\underset{\cdot}{k}, l, m\}^{(a)}$ and the $S\{\underset{\cdot}{k}, l, n\}^{(b)}$ into a $S\{\underset{\cdot}{k}, l, mn\}^{(c)}$ and then exhibit this as a $SGS\{\underset{\cdot}{k}, l, mn\}^{(c)}$. In fact every $\underset{\cdot}{k}$-id$^{(c)}$ of the $S\{\underset{\cdot}{k}, l, mn\}^{(c)}$ lies in and determines a set $s_g^{(c)} \equiv S\{k, g, mn\}^{(c)}$ $(g = 1 \ldots . l)$: for the $\underset{\cdot}{k}$-id$^{(c)}$ is compounded from a certain $\underset{\cdot}{k}$-id$^{(a)}$ and a certain $\underset{\cdot}{k}$-id$^{(b)}$; the $\underset{\cdot}{k}$-id$^{(a)}$ lies in and determines a $s_g^{(a)} \equiv S\{\underset{\cdot}{k}, g, m\}^{(a)}$ of the $SGS\{\underset{\cdot}{k}, l, m\}^{(a)}$; the $\underset{\cdot}{k}$-id$^{(b)}$ lies in and determines a $s_g^{(b)} \equiv S\{\underset{\cdot}{k}, g, n\}^{(b)}$ of the $SGS\{\underset{\cdot}{k}, l, n\}^{(b)}$; these $S\{\underset{\cdot}{k}, g, m\}^{(a)}$, $S\{\underset{\cdot}{k}, g, n\}^{(b)}$ compound within the $S\{\underset{\cdot}{k}, l, mn\}^{(c)}$ uniquely into a $S\{\underset{\cdot}{k}, g, mn\}^{(c)}$; this $S\{\underset{\cdot}{k}, g, mn\}^{(c)}$ is determined in this way by any $\underset{\cdot}{k}$-id$^{(c)}$ lying in it.

(c). *Given a two-fold $SGS\{\underset{\cdot}{k}, 2, m\}^{(a)}\{s_{gh}^{(a)}\}$ $(g, h = 1 \ldots . m)$ and a two-fold $SGS\{\underset{\cdot}{k}, 2, n\}^{(b)}\{s_{ij}^{(b)}\}$ $(i, j = 1 \ldots . n)$: to construct a two-fold $SGS\{\underset{\cdot}{k}, 2, mn\}^{(c)}$ $\{s_{tu}^{(c)}\}$ $(t, u = 1 \ldots . mn)$.*

We establish arbitrarily a 1—1 correspondence between the mn indices t and the mn pairs of indices g, i, and again arbitrarily a 1—1 correspondence between the mn indices u and the mn pairs of indices h, j. We compound the $\underset{\cdot}{k}$-id$^{(a)}$ $s_{gh}^{(a)}$ and the $\underset{\cdot}{k}$-id$^{(b)}$ $s_{ij}^{(b)}$ into a $\underset{\cdot}{k}$-id$^{(c)}$, say $s_{tu}^{(c)}$, where t corresponds to g, i, and u corresponds to h, j. The matricular arrangement $\{s_{tu}^{(c)}\}$ is then obviously a two-fold $SGS\{\underset{\cdot}{k}, 2, m\}^{(c)}$.

(d). *Given a* $S\{k,\ 2,\ m\}^{(a)}$ *and a* $S\{k,\ 2,\ n\}^{(b)}$: *to construct a* $S\{k,\ 2,\ mn\}^{(c)}$.

We extend the two given k-idic systems to $S\{\underset{\cdot}{k},\ 2,\ m\}^{(a)}$, $S\{\underset{\cdot}{k},\ 2,\ n\}^{(b)}$(§15 c), and compound these two $\underset{\cdot}{k}$-idic systems into a $\underset{\cdot}{k}$-idic system $S\{\underset{\cdot}{k},\ 2,\ mn\}^{(c)}$ (a). This $S\{\underset{\cdot}{k},\ 2,\ mn\}^{(c)}$ contains besides the mn $\underset{\cdot}{k}$-idic sequences $\{c_{ij} \ldots c_{ij}\}\binom{i = 1 \ldots m}{j = 1 \ldots n}$, exactly the $S\{k,\ 2,\ mn\}^{(c)}$ desired.

(e). *Given a* $SGS\{k,\ 2,\ m\}^{(a)}$ *and a* $SGS\{k,\ 2,\ n\}^{(b)}$: *to construct a* $SGS\{k,\ 2,\ mn\}^{(c)}$.

The $S\{k,\ 2,\ mn\}^{(c)}$ of (d) may in this case be exhibited as a $SGS\{k,\ 2,\ mn\}^{(c)}$, since the mn sequences $\{c_{ij} \ldots\ldots c_{ij}\}$ omitted constitute one set $s_1 \equiv S\{\underset{\cdot}{k},\ 1,\ mn\}$ of the $SGS\{\underset{\cdot}{k},\ 2,\ mn\}^{(c)}$ (b).

In III 12 I give a direct construction of a $SGS\{4,\ 2,\ 4^n\}$.

(f). *Given a* $S[k,\ l,\ m]^{(a)}$ *and a* $S\{k,\ l,\ k\}^{(b)}$: *to construct a* $S\{k,\ l,\ m\}^{(a)}$.

The $S[k,\ l,\ m]^{(a)}$ contains m_l/k_l k-ads$^{(a)}$. We take every k-ad$^{(a)}$ $[a_{i_1} \ldots a_{i_f} \ldots a_{i_k}]$, *each in any certain way* as a k-id$^{(a)}$ $\{a_{i_1} \ldots\ldots a_{i_f} \ldots\ldots a_{i_k}\}$. The k_l k-ids of the $S\{k,\ l,\ k\}^{(b)}$ are obtained from the k-id $\{b_1 \ldots\ldots b_f \ldots\ldots b_k\}$ by certain permutations j of the positions of the letters: $\{b_{j_1} \ldots\ldots b_{j_f} \ldots\ldots b_{j_k}\} = \{b_1 \ldots\ldots b_f \ldots\ldots b_k\}_j$. The position-permutation j changes $\{a_{i_1} \ldots\ldots a_{i_f} \ldots\ldots a_{i_k}\}$ into $\{a_{ij_1} \ldots\ldots a_{ij_f} \ldots\ldots a_{ij_k}\}$ $= \{a_{i_1} \ldots\ldots a_{i_f} \ldots\ldots a_{i_k}\}_j$. We obtain thus in all m_l k-ids$^{(a)}$; they constitute a $S\{k,\ l,\ m\}^{(a)}$.

We speak of the *compositions* of the k-ad $[a_{i_1} \ldots\ldots a_{i_f} \ldots\ldots a_{i_k}]$ and the k-id $\{b_{j_1} \ldots\ldots b_{j_f} \ldots\ldots b_{j_k}\}$ into the k-id $\{a_{ij_1} \ldots\ldots a_{ij_f} \ldots\ldots a_{ij_k}\}$ and of the $S[k,\ l,\ m]^{(a)}$ and the $S\{k,\ l,\ k\}^{(b)}$ into the $S\{k,\ l,\ m\}^{(a)}$. These composition-processes are obviously not unique.

(g). *Given a* $SGS[k,\ l,\ m]^{(a)}$ *and a* $SGS\{k,\ l,\ k\}^{(b)}$: *to construct a* $SGS\{k,\ l,\ m\}^{(a)}$.

We compound by (f) the $S[k,\ l,\ m]^{(a)}$ and the $S\{k,\ l,\ k\}^{(b)}$ into a $S\{k,\ l,\ m\}^{(a)}$ and then exhibit this as a $SGS\{k,\ l,\ m\}^{(a)}$. In fact every k-id$^{(a)}$ of the $S\{k,\ l,\ m\}^{(a)}$ lies in and determines a set $s_g^{(a)} \equiv S\{k,\ g,\ m\}^{(a)}$ $(g = 1 \ldots\ldots l)$: for the k-id$^{(a)}$ of the $S\{k,\ l,\ m\}^{(a)}$ arises by composition of a certain k-ad$^{(a)}$ and a certain k-id$^{(b)}$; the k-ad$^{(a)}$ lies in and determines a $s_g^{(a)} \equiv S[k,\ g,\ m]^{(a)}$ of the $SGS[k,\ l,\ m]^{(a)}$; the k-id$^{(b)}$ lies in and determines a $s_g^{(b)} \equiv S\{k,\ g,\ k\}^{(b)}$ of the $SGS\{k,\ l,\ k\}^{(b)}$; these $S[k,\ g,\ m]^{(a)}$, $S\{k,\ g,\ k\}^{(b)}$ compound within the $S\{k,\ l,\ m\}^{(a)}$ uniquely into a $S\{k,\ g,\ m\}^{(a)}$; this $S\{k,\ g,\ m\}^{(a)}$ is determined in this way by any k-id$^{(a)}$ lying in it.

(h). *Given a* $S[k,\ 2,\ m]^{(a)}$ *and a* $S[k,\ 2,\ n]^{(b)}$ *and an exactly doubly transitive* $G^k_{k_2}$: *to construct a* $S[k,\ 2,\ mn]^{(c)}$ ($k=p^t$ (§11 a)).

The $G^k_{k_2}$ determines (§11 a) a $S\{k,\ 2,\ k\}^{(d)}$ whose k_2 position-permutations j (see f) form a group, viz. the $G^k_{k_2}$. We compound by (f) the $S[k,\ 2, m]^{(a)}$, $S\{k,\ 2,\ k\}^{(d)}$ into a $S\{k,\ 2,\ m\}^{(a)}$, and similarly compound the $S[k,\ 2, n]^{(b)}$, $S\{k, 2,\ k\}^{(d)}$ into a $S\{k,\ 2,\ n\}^{(b)}$. We compound by (d) the $S\{k,\ 2,\ m\}^{(a)}$, $S\{k,\ 2,\ n\}^{(b)}$ into a $S\{k,\ 2,\ mn\}^{(c)}$. Any k-id$^{(c)}$ of this $S\{k, 2, mn\}^{(c)}$ by the k_2 position-permutations j leads to k_2 k-ids$^{(c)}$; *since the* k_2 *position-permutations form a group*, these k_2 k-ids$^{(c)}$ are found in the $S\{k,\ 2,\ mn\}^{(c)}$.

We may replace these k_2 k-ids$^{(c)}$ by one k-ad$^{(c)}$. The $S\{k,\ 2,\ mn\}^{(c)}$ becomes by this decomposition with respect to the $S\{k,\ 2, k\}^{(d)}$ (which is the reverse of the composition (f)) the $S[k,\ 2,\ mn]^{(c)}$ desired.

(i). *Given a* $SGS[k,\ 2,\ m]^{(a)}$ *and a* $SGS[k,\ 2,\ n]^{(b)}$ *and an exactly doubly transitive* $G^k_{k_2}$: *to construct a* $SGS[k,\ 2,\ mn]^{(c)}$ ($k=p^t$ (§11 a b)).

The $G^k_{k_2}$ determines (§11 b) a $SGS\{k,\ 2,\ k\}^{(d)}$. We obtain by (g) $SGS\{k, 2, m\}^{(a)}$, $SGS\{k,\ 2, n\}^{(b)}$, and then by (d) a $SGS\{k,\ 2, mn\}^{(c)}$, which, in view of the group-origin of the $SGS\{k,\ 2,\ k\}^{(d)}$, decomposes with respect to the $SGS\{k,\ 2, k\}^{(d)}$ into the $SGS[k,\ 2,\ mn]^{(c)}$ desired.

In III 12 I give a direct construction of a $SGS[4,\ 2,\ 4^n]$.

(j). *To construct a* $SGS\{k,\ 2,\ m\}$ *where* m *is any integer with the different primes* $p_1 \ldots . p_t$, $m=\prod_{i=1}^{t} p_i^{h_i}$, *and where* k *is any integer subject only to the conditions* $k \leqq p_i^h$ $(i=1 \ldots . t)$, $k \geqq 2$.

We have by §11 b (aδ) $SGS\{p_i^{h_i},\ 2, p_i^{h_i}\}$ $(i=1 \ldots . t)$, and so by §9 e $SGS\{k,\ 2,\ p_i^{h_i}\}$ $(i=1 \ldots . t)$, and so by composition (d) the $SGS\{k,\ 2,\ m\}$ desired.

I give a few other applications of the composition theorems (a . . . . i).

(k). *To construct a two-fold* $SGS\{k,\ 2,\ m\}$ *where* $m=\prod_{i=1}^{t} p_i^{h_i}$, *where* m *if divisible by* 2 *is divisible by* 4, *and where* k *is subject only to the conditions* $k \leqq p_i^{h_i}-1$ $(i=1 \ldots . t)$, $k \geqq 2$.

We have by §15 l two-fold $SGS\{p_i^{h_i}-1,\ 2, p_i^{h_i}\}$ $(i=1 \ldots . t)$, and so by §13 f two-fold $SGS\{k,\ 2, p_i^{h_i}\}$ $(i=1 \ldots . t)$, and so by composition (c) the two-fold $SGS\{k,\ 2,\ m\}$ desired.

(l). *To construct a* $SGS\{\underset{\cdot}{k}, l, m\}$ *where* $m = \prod_{i=1}^{t} p_i^{h_i}$ *and where* k *and* l *are subject only to the conditions* $l \leqq k \leqq p_i^{h_i}$ $(i = 1 \ldots . t)$.

We have by §15 p $SGS\{\underset{\cdot}{p_i^{h_i}}, p_i^{h_i}, p_i^{h_i}\}$ $(i = 1 \ldots . t)$, and so by §13 d $SGS\{\underset{\cdot}{k}, k, p_i^{h_i}\}$ $(i = 1 \ldots . t)$, and so (by the definition of a $SGS\{\underset{\cdot}{k'}, l', m'\}$) $SGS\{\underset{\cdot}{k}, l, p_i^{h_i}\}$ $(i = 1 \ldots . t)$, and so by composition (b) the $SGS\{\underset{\cdot}{k}, l, m\}$ desired.

(m). *Given a two-fold* $SGS\{\underset{\cdot}{2}, 2, m\}^{(a)}\{s_{ij}^{(a)}\}$ *and a two-fold* $SGS\{\underset{\cdot}{2}, 2, m\}^{(b)}$ $\{t_{xy}^{(b)}\}$: *to construct a two-fold* $SGS\{\underset{\cdot}{3}, 3, m\}^{(a)}$. [By (k) this gives us $SGS\{\underset{\cdot}{3}, 3, m\}$ for m any integer $\geqq 3$ odd or evenly even.]

The two-fold $SGS\{\underset{\cdot}{2}, 2, m\}^{(a)}\{s_{ij}^{(a)}\}$ is

$$s_{ij}^{(a)} = \{a_{f_{ij}}\, a_{g_{ij}}\}\,; \qquad (i, j = 1 \ldots . m)$$

the i^{th} row α_i' is a $S\{\underset{\cdot}{2}, 1, m\}^{(a)}$, and likewise the j^{th} column α_j''; hence each of the indices f_{ij}, g_{ij} effects for i fixed or for j fixed a substitution on the m indices $j = 1 \ldots . m$ or $i = 1 \ldots . m$; and further, every $\underset{\cdot}{2}$-id $\{a_f a_g\}$ appears once as an $s_{ij}^{(a)}$,

$$\{a_f\, a_g\} = s_{i_{fg} j_{fg}}^{(a)}. \qquad (f, g = 1 \ldots . m)$$

We make similar notations and remarks in connection with the two-fold $SGS\{\underset{\cdot}{2}, 2, m\}^{(b)}$ $\{t_{xy}^{(b)}\}$,

$$t_{xy}^{(b)} = \{b_{h_{xy}}\, b_{k_{xy}}\}, \ \{b_h\, b_k\} = t_{x_{hk} y_{hk}}^{(b)}. \quad (x, y;\, h, k = 1 \ldots . m)$$

Then the $SGS\{\underset{\cdot}{3}, 3, m\}^{(a)}$ desired is the arrangement $\{u_{xyj}^{(a)}\}$ of m^3 $\underset{\cdot}{3}$-ids,

$$u_{xyj}^{(a)} = \{a_{f_{h_{xy}j}}\, a_{g_{h_{xy}j}}\, a_{k_{xy}}\}. \qquad (x, y, j = 1 \ldots . m)$$

We speak of the x^{th} row and the y^{th} column of this $\{u_{xyj}^{(a)}\}$. Writing $u_{xy*}^{(a)} = \{u_{xy1}^{(a)}, \ldots . u_{xyj}^{(a)}, \ldots . u_{xym}^{(a)}\} = \{\alpha_{h_{xy}}'\, a_{k_{xy}}\}$ we see that the $\{u_{xyj}^{(a)}\}$ results from a composition in a certain way of the given $\{s_{ij}^{(a)}\}$ $\{t_{xy}^{(b)}\}$. Obviously this composition can be effected in various ways distinct from but similar to the way chosen above.

Identifying the $\underset{\cdot}{3}$-id $u_{xyj}^{(a)}$ with the $\underset{\cdot}{3}$-id $\{a_c\, a_d\, a_e\}$,

$$c = f_{h_{xy}j}, \ d = g_{h_{xy}j}, \ e = k_{xy},$$

we find easily that of the six indices $cdexyj$ any three (except cdj and xye) with values given at will determine the remaining three uniquely.

Hence the m^3 $\underset{\bullet}{3}$-ids $u^{(a)}_{xyj}$ (x, y, $j = 1 \ldots . m$) form the $S\{\underset{\bullet}{3}, 3, m\}^{(a)}$ $u^{(a)}_{***} = \{u^{(a)}_{xyj}\}$. Further, the m $\underset{\bullet}{3}$-ids $u^{(a)}_{xyj}$ ($y = 1 \ldots . m$) form a $S\{\underset{\bullet}{3}, 1, m\}^{(a)}$ $u^{(a)}_{x*j}$, and the m^2 $\underset{\bullet}{3}$-ids $u^{(a)}_{xyj}$ ($y, j = 1 \ldots . m$), that is, the m $u^{(a)}_{x*j}$ ($j = 1 \ldots . m$) form a $S\{\underset{\bullet}{3}, 2, m\}^{(a)}$ $u^{(a)}_{x**}$, while the m $u^{(a)}_{x**}$ ($x = 1 \ldots . m$) form the $u^{(a)}_{***} = \{u^{(a)}_{xyj}\}$. Thus the $\{u^{(a)}_{xyj}\}$ is a $SGS\{\underset{\bullet}{3}, 3, m\}^{(a)}$ in this way by rows, and also similarly by columns.

(n). Two-fold $SGS\{\underset{\bullet}{3}, 3, 3\}$. The two-fold $SGS\{\underset{\bullet}{2}, 2, 3\}$ of §15 d yields by proper application of (m) the following two arrangements $\{u_{xyj}\}$, each a two-fold $SGS\{\underset{\bullet}{3}, 3, 3\}$:

aa	*a*	*bc*	*b*	*cb*	*c*
bb		*ca*		*ac*	
cc		*ab*		*ba*	
bc	*c*	*cb*	*a*	*aa*	*b*
ca		*ac*		*bb*	
ab		*ba*		*cc*	
cb	*b*	*aa*	*c*	*bc*	*a*
ac		*bb*		*ca*	
ba		*cc*		*ab*	

aa	*a*	*bb*	*b*	*cc*	*c*
bc		*ca*		*ab*	
cb		*ac*		*ba*	
bb	*c*	*cc*	*a*	*aa*	*b*
ca		*ab*		*bc*	
ac		*ba*		*cb*	
cc	*b*	*aa*	*c*	*bb*	*a*
ab		*bc*		*ca*	
ba		*cb*		*ac*	

The second of these taken by rows gives the $SGS\{\underset{\bullet}{3}, 3, 3\}$ found in §15 g.

III.

WHIST-TOURNAMENT ARRANGEMENTS *Wh* [*m*].

1. *Wh* [*m*]. The m players ($m \equiv 0$, mod. 4) in a whist tournament are designated by the m letters $a_1, a_2, \ldots\ldots, a_m$. The players play seated by fours at $\frac{1}{4}m$ tables. Each table stands according to the points of the compass, and the players sit accordingly. $\{a_i a_j a_k a_l\}$ occupy the respective positions $\{N S W E\}$. a_i and a_j play together as partners against their opponents, the partners a_k and a_l. A say *evening* is any arrangement of the m players at the $\frac{1}{4}m$ tables.

In arranging the programme for a whist tournament different whist-considerations lead to different desiderata. The problem in every case is in ultimate formulation purely tactical.

Mr. Mitchell, in his *Duplicate Whist* (p. 17), formulates a problem which has considerable tactical interest. We define—